The University of Sydney
School of Mathematics and Statistics

Integrable Systems

MATH4403: Applied Mathematics Honours C — Semester 2, 2017

Lecturers: Nalini Joshi (Room 629, Carslaw Building), Milena Radnović (Room 633, Carslaw Building), Yang Shi (Room 633, Carslaw Building)

Consultation: By appointment - please arrange via e-mail to either `nalini.joshi@sydney.edu.au`, `milena.radnovic@sydney.edu.au`, or `yang.shi@sydney.edu.au`

Course web-page: `http://www.maths.usyd.edu.au/u/UG/HM/AMH2/`

Outline: The mathematical theory of integrable systems has been described as one of the most profound advances of twentieth century mathematics. They lie at the boundary of mathematics and physics and were discovered through a famous paradox that arises in a model devised to describe the thermal properties of metals (called the Fermi-Pasta-Ulam paradox).

In attempting to resolve this paradox, Kruskal and Zabusky discovered exceptional properties in the solutions of a non-linear PDE, called the Korteweg-de Vries equation (KdV). These properties showed that although the solutions are waves, they interact with each other as though they were particles, i.e., without losing their shape or speed, until then thought to be impossible for solutions of non-linear PDEs. Kruskal invented the name *solitons* for these solutions.

Solitons are known to arise in other non-linear PDEs and also in partial difference equations. These systems and their symmetry reductions are now called "integrable systems". These systems occur as universal limiting models in many physical situations.

This course introduces the mathematical properties of such systems. In particular, we will study their solutions, symmetry reductions called the Painlevé equations and their discrete versions. It focuses on mathematical methods created to describe the solutions of such equations and their interrelationships.

Course Objectives and Outcomes:

- Understand the *inverse scattering transform method*: how to use it to solve integrable systems and find solitons; how to prove that it works for certain initial conditions.
- Understand the transformation theory that relates integrable systems to each other and the reductions from PDEs to ODEs.
- Understand how to use symmetry groups to describe transformations, find special solutions, recurrence relations and related discrete integrable systems.
- Describe other properties of solutions of integrable systems, in particular behaviours that occur in limits.

Assessment and Exam:

- Assessment will be based on weekly exercises, which will be released progressively on the class webpage at `http://www.maths.usyd.edu.au/u/UG/HM/AMH2/` during weeks 2-11. Solution of each exercise will be due at the first lecture in the subsequent week. Each exercise will be marked out of 6. Full marks for 10 exercises will total 60% of the final mark.

- There will be a take-home exam worth 40% at the end of the semester.
- Your answers to each exercise and the take-home exam are required to be submitted to `Turnitin` through `Blackboard` (i.e., the University's Learning Management System), as the University has mandated the use of this text-based similarity detecting software for all text-based written assignments.

References and Supporting Material: Useful texts include

- M. J. Ablowitz and H. Segur, *Solitons and the inverse scattering transform* SIAM, Philadelphia, USA, 1981.
- M. J. Ablowitz and P.A. Clarkson, *Solitons, nonlinear evolution equations and inverse scattering* Cambridge University Press, Cambridge, UK, 1991.
- P.G. Drazin and R.S. Johnson, *Solitons : an introduction* Cambridge University Press, Cambridge, UK, 1989.
- M. Noumi, *Painlevé equations through symmetry*, American Mathematical Society, Providence, R.I., USA, 2004.

Interesting Links: There are many interesting links on solitons, provided on the course webpage. Have a look at

- An account of John Scott Russell's discovery of "that singular and beautiful phenomenon, which I have called the wave of translation."
- A modern attempt by mathematicians to recreate Scott Russell's wave in the Union Canal near Edinburgh.
- The Wikipedia page on Solitons (whose first and second definitions are still not quite right).
- The Wikipedia page on the Korteweg-de Vries equation.