Subject Information Guide

Model Theory (MAT4ATA)

Semester 1, 2016

Administration and contact details

Host Department	**Department of Mathematics and Statistics**
Host Institution	**La Trobe University**.
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Subject details

Handbook entry URL	Same as below
Subject homepage URL	http://tomasz-kowalski.ltumathstats.com/model-theory
Honours student hand-out URL	Same as above
Start date:	3 March 2016
End date:	26 May 2016
Contact hours per week:	2 hour lecture per week
Lecture day and time:	Thursday, 2pm-4pm
Description of electronic access arrangements for students (for example, WebCT)	

Subject content

1. Subject content description

The subject is an introduction to model theory: a branch of mathematics that deals with classification of structures by means of logical formulas. Mathematical structures (groups, rings, fields, ordered sets, lattices, whatever) can be classified according to which logical formulas are true in them. Conversely, given some logical formulas, one wants to find classes of structures in which the formulas are true. If a formula F is true in a structure **S**, we say that **S** is a *model* of F, hence, *model theory*. Mathematicians have used models all the time, they just couldn't be bothered to build a theory around it. For example, to demonstrate that the fifth Euclid's postulate does not follow from the rest, one can take a sphere: a (non-standard) model of all the postulates except the fifth. Model *theory* however is relatively young, it did not exist befere the second half of the 20th Century.

We begin with a rather quick overview of first-order logic: enough of it to be able to present Henkin's completeness proof. We spend some time discussing *compactness,* a property that says: if something follows from an infinite set of assumptions it also follows from a finite one. Surprisingly, it has rather striking consequences, such as the fact that one cannot characterise finite structures by first-order formulas.

Then, we deal with basic operations on models: homomorphisms, substructures, direct products, unions of chains. We tie these constructions with logical formulas they preserve by means of some *preservation theorems.* For example, a formula is preserved by substructures if and only if it is logically equivalent to a formula "for all x,y,z,… F(x,y,z,...)". On the way, we introduce and important tool: *diagrams*. We also briefly encounter *elementary equivalence.* Structures **A** and **B** are elementarily equivalent if they satisfy precisely the same first order sentences. Intuitively, **A** and **B** are indistinguishable by first-order properties. Elementary equivalence is weaker than isomorphism (of course); in fact, strictly weaker (not quite of course).

Quantifier elimination comes next. As the name suggests, it is a method of getting rid of quantifiers from sentences. If this is possible for some class **K** of structures, then the first-order theory of **K** can be greatly simplified: often to the point of being *decidable*. There are other uses, too, and in fact quantifier elimination is the only branch of model theory that has produced extensive industrial applications. Quantifier elimination is connected with the property of *model completeness,* which we will briefly mention, mostly because we need it in a model theoretic proof of a version of Hilbert's Nullstellensatz.

Then, we look at the *ultraproduct* construction. An ultraproduct is a special quotient of a direct product; the construction is fundamental in model theory because it preserves all first-order formulas. We connect ultraproducts with compactness on the one hand, and with elementary equivalence on the other. We give some examples of non-standard models that can be obtained via ultraproducts, among them infinitely large "natural numbers" and infinitely small "reals". Getting well acquainted with ultraproducts is one of the main goals of the subject.

Ultraproducts are very good at deling with elementary equivalence, but there is a whole family of equivalences intermediate between isomorphism and elementary equivalence. These are defined and analysed by means of *Ehrenfeucht-Fraïssé games* played on structures. The players are the universal quantifier (Abelard) and the existential quantifier (Eloise). Eloise has a winning strategy for a game of length n if n-generated substructures chosen during the play are isomorphic. We will mostly focus on games of countably infinite length and the associated notion of *back-and-forth equivalence*. Proofs via games can be messy in written form (because you need to describe a winning strategy), but finding a proof is just playing the game cleverly, so it is good fun. Proficiency at these games is another main goal.

Our final topics are *homogeneity* and *omega-categoricity.* A structure **S** is *homogeneous* if every isomorphism between finitely generated substructures of **S** extends to an automorphism of **S**. A good example of such an **S** is the random graph. Omega-categoricity is a property of sets of sentences (theories): a theory T is *omega-categorical* if all countable models of T are isomorphic. A structure **S** is called omega-categorical is its first-order theory is. An important method of constructing homogeneous and omega-categorical structures is that of *Fraïssé limit.* A good example of what is involved is the following. Consider the integers and the rationals as ordered sets. Both have the property that every finite linearly ordered set embeds in them. But one is homogeneous (the rationals) and the other is not. Fraïssé's construction applied to the set of all finite linear order produces exactly the rationals.

2. Week-by-week topic overview

Weeks 1-2: First-order logic, syntax and semantics. Compactness.
Week 3-4: Models, homomorphisms, embeddings, direct products. Elementary substructures, elementary equivalence. Diagrams, preservation theorems.
Weeks 5-6: Quantifier elimination. Model completeness.
Weeks 7-8: Ultraproducts, compactness again, elementary equivalence again.
Weeks 9-10: Back-and-forth arguments, Ehrenfeucht-Fraïssé games.
Weeks 11-12: Homogeneity, omega-categoricity and Fraïssé limit construction.

3. Assumed prerequisite knowledge and capabilities

The subject is an introductory one, so there are no specific prerequisites. However, some basic knowledge of classical algebra and discrete mathematics (groups, rings, fields, graphs, ordered sets) will be assumed.

4. Learning outcomes and objectives

- Familiarity with basic tools and techniques of model theory.
- Ability to apply model theoretic tools, especially ultraproducts and Ehrenfeucht-Fraïssé games in a variety of contexts.

AQF specific Program Learning Outcomes and Learning Outcome Descriptors (if available):

AQF Program Learning Outcomes addressed in this subject	**Associated AQF Learning Outcome Descriptors for this subject**
Familiarity with basic tools and techniques of model theory.	K1, K2
Ability to apply model theoretic tools, especially ultraproducts and Ehrenfeucht-Fraïssé games in a variety of contexts.	S1, S2, S3, A2

Learning Outcome Descriptors at AQF Level 8
Knowledge
K1: coherent and advanced knowledge of the underlying principles and concepts in one or more disciplines
K2: knowledge of research principles and methods
Skills
S1: cognitive skills to review, analyse, consolidate and synthesise knowledge to identify and provide solutions to complex problem with intellectual independence
S2: cognitive and technical skills to demonstrate a broad understanding of a body of knowledge and theoretical concepts with advanced understanding in some areas
S3: cognitive skills to exercise critical thinking and judgement in developing new understanding
S4: technical skills to design and use in a research project
S5: communication skills to present clear and coherent exposition of knowledge and ideas to a variety of audiences
Application of Knowledge and Skills
A1: with initiative and judgement in professional practice and/or scholarship
A2: to adapt knowledge and skills in diverse contexts
A3: with responsibility and accountability for own learning and practice and in collaboration with others within broad parameters
A4: to plan and execute project work and/or a piece of research and scholarship with some independence

5. Learning resources

Main textbook:

- Wilfrid Hodges, *Shorter Model Theory.*

Recommended:

- David Marker, *Model Theory. An Introduction.*
- Martin Goldstern, Haim Judah, *The Incompleteness Phenomenon. A New Course in Mathematical Logic.*

Course notes will be available from the subject's web page:
http://tomasz-kowalski.ltumathstats.com/model-theory

Slides of lectures as well as assignments will also be available there in due course.

6. Assessment

Exam/assignment/classwork breakdown					
Exam	40 %	**Assignment**	60 %	**Class work**	
Assignment due dates		Assignment 1 Assignment 2 Assignment 3	Week 4 Week 8 Week 12		
Approximate exam date				Take home exam: 1 June 2016.	

Institution Honours program details

Weight of subject in total honours assessment at host department	
Thesis/subject split at host department	
Honours grade ranges at host department:	
H1	80-100 %
H2a	70-79 %
H2b	60-69 %
H3	50-59 %